Une formulation de type Vlassov-Poisson pour les equations d'Euler des fluides parfaits incompressibles

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A VLASOV-POISSON TYPE FORMULATION OF THE EULER EQUATIONS FOR PERFECT INCOMPRESSIBLE FLUIDS

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Résumé

On introduit une nouvelle formulation, "cinétique", des équations d'Euler des fluides parfaits incompressibles, comme limite singulière du système de Vlassov-Poisson, bien connu en Physique des Plasmas. On montre des exemples et des calculs numériques de solutions généralisées "cinétiques" des équations d'Euler.

Abstract

A kinetic formulation is introduced for the Euler equations of incompressible perfect fluids, that can be seen as the singular limit of the Vlasov-Poisson equations, well known in Plasma Physics. Examples and numerical calculations of generalized kinetic solutions to the Euler equations are shown.

Mots-clés

Equations cinétiques, Vlassov, Euler.

Keywords

Kinetic equations, Euler, Vlasov.

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A "kinetic" formulation is introduced for the Euler equations of incompressible perfect fluids : Find a "distribution" function $f = f(t,x,v)$ (where t,x,v are the time, space and velocity variables) and a potential force field $E = E(t,x) = -\,\mathrm{grad}_x\phi(t,x)$, such that :

$$\int f dv \equiv 1 \quad \text{and} \quad \partial_t f + \mathrm{div}_x(vf) + \mathrm{div}_x(Ef) = 0$$

in the sense of distributions. By the first equation, the incompressibility is expressed (when the fluid is homogeneous), by the second one it is said that E accelerates the material particles.

Classical solutions of the Euler equations correspond to delta distribution function : $f(t,x,v) = \delta(v-V(t,x))$, where V is the classical velocity field. The kinetic formulation can be seen as the singular limit of the Vlasov-Poisson system, where the potential ϕ is linked to the distribution function by a Poisson equation, e.g. $\alpha\, \mathrm{div}_x E = \int f dv - 1$. This would correspond (when α goes to zero) to a "quasineutrality" assumption in Plasma Physics.

The new kinetic formulation does not lead to global existence or uniqueness results but provides a nice framework to the study of sequences of solutions to the Euler equations, when the second derivatives of the pressure field are uniformly bounded.

Various examples of generalized kinetic solutions to the Euler equations are considered. Numerical calculations are shown.

0 - A comparison of Vlasov-Poisson and Euler equations

It is a traditional task of Statistical Mechanics to derive macroscopic models, as the Fluid Mechanics equations, from microscopic models, as the kinetic equations. For example, it has been known for a long time that one can formally derive the Euler equations of compressible perfect gaz from the Boltzmann equations when the mean free path vanishes. In the recent years,

important steps have been made towards a rigorous mathematical theory of the kinetic equations and their limits when various significant physical parameters vanish [5,8,2...].

Nevertheless, to the best of our knowledge, it has been ignored that a direct link can be established between the Vlasov-Poisson system in Plasma Physics (where the particle interactions are due to their electrical charges and not to their collisions) and the Euler equations of perfect incompressible fluids in Fluid Mechanics.

In our opinion, this ignorance is partly due to the absence of clear physical link between these models, and partly due to the conventional writing of these equations that almost completely hides their link. The first goal of our paper is to show how close are the Vlasov-Poisson system and the Euler equations, provided their Lagrangian description in terms of particle trajectories is considered.

Thank to this new link with the Vlasov-Poisson system, a "kinetic" formulation of the incompressible Euler equations is introduced in this paper. The main advantage of the new formulation is to take place half way between the conventional strong formulation of the Euler equations and the extremely weak formulation recently introduced by Diperna and Majda [6] with, their concept of "measure-valued" (mv) solutions. Unfortunately, the new formulation does not provide any global existence theorem so far. This is mainly due to the lack of any reasonable estimate on the pressure field. Moreover, as it will be shown, the kinetic formulation of the Euler equations can be seen as the (very !) singular limit of the Vlasov-Poisson equations, for which global existence theorems have been obtained only very recently by Diperna and Lions with the help of recent compactness results on transport equations [5,8].

Nevertheless, the new formulation, introduced in the first section, provides a very natural framework to the analysis of sequences of solutions to the Euler equations when a uniform control on the pressure field is a priori known. It will be established, in the second section, that the limit can be easily described as a generalized "kinetic" solution to the Euler equations. Moreover, in that case, the kinetic formulation is completely consistent with the concept of generalized incompressible flows satisfying a generalized least action principle, that has been recently introduced [3].

In the third section, two explicit examples will be discussed showing the behaviour predicted in the second section. Though no existence theorem will be obtained in this paper, some explicit non classical "Kinetic" solutions to the Euler equations will be presented in the fourth section, and numerical calculations done with the help of "particle methods" will be shown in section 5.

I - The Kinetic formulation of Euler's equations and their link with the Vlasov-Poisson system

For simplicity, the physical space will be the d-dimensional torus : $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ for d = 2 or 3. Let us consider a smooth homogeneous incompressible perfect fluid flow on $\mathbb{T}^d$. Traditionaly, the fluid motion is described in terms of the velocity field $v = v(t,x)$, $t \in \mathbb{R}$, $x \in \mathbb{T}^d$ and the pressure field $\phi = \phi(t,x)$. After suitable normalizations, one gets, for a homogeneous incompressible perfect fluids, the following equations :

(1.1) $\quad \operatorname{div} v = 0$ (incompressibility condition)

(1.2) $\quad \partial_t v + \operatorname{div}(v \otimes v) + \operatorname{grad}\phi = 0$

or, equivalently (if v is smooth) :

(1.2bis) $\quad \partial_t v + (v.\operatorname{grad})v + \operatorname{grad}\phi = 0.$

A less traditional viewpoint is provided by the Lagrangian description in terms of material particles. Indeed, behind equation (1.2bis), one recovers the classical Newton law :

(1.3) $\quad \ddot{x} = E(t,x)$

where $x = x(t)$ denotes any particle path and E the force field :

(1.4) $\quad E(t,x) = - \operatorname{grad}_x \phi(t,x).$

For a homogenous fluid, the incompressibility condition (1.1) exactly means that the density of particles is uniform in space and time :

(1.5) $\quad \rho(t,x) \equiv 1$

Let us now look at the Vlasov-Poisson system, in the particular case when the motion of electrons in presence of non moving neutrons is considered. Up to suitable normalizations, one gets for the "distribution" function $f = f(t,x,v)$, $t \in \mathbb{R}$, $x \in \mathbb{T}^d$, $v \in \mathbb{R}^d$ (that gives the probability of finding a particle located at time t and position x with velocity v), the Vlasov equation :

(1.6) $\quad \partial_t f + \operatorname{div}_x(vf) + \operatorname{div}_v(Ef) = 0$

where E denotes the electric field :

(1.7) $E(t,x) = -\,\mathrm{grad}_x\phi(t,x)$

and ϕ is the electric potential. ϕ is linked to f through the Poisson equation :

(1.8) $\alpha\Delta_x\phi = \int f(t,x,v)\,dv - 1$ E periodic in x,

where α is a physical constant and 1 is the neutron density. The integral of f with respect to v is nothing but the density of particles at time t and position :

(1.9) $\rho(t,x) = \int f(t,x,v)\,dv.$

Therefore, (1.8) can be rewritten as :

(1.10) $\rho(t,x) = 1 + \alpha\Delta_x\phi(t,x).$

Moreover, behind (1.6), one recognizes Newtons's law that says that the particles are accelerated by the force field E :

(1.11) $\ddot{x} = E(t,x)$

where $x = x(t)$ denotes a typical particle path.

Indeed equation (1.6) is nothing but the Liouville equation in the phase space (x,v) for the Hamiltonian ordinary differential system :

(1.12) $\dot{x} = v,\ \dot{v} = E(t,x)$

equivalent to (1.11). In other words, the distribution function of any system of particles obeying (1.11) is governed by (1.6).

Thus, the Vlasov-Poisson system (1.6-7-8) is equivalent to (1.7-10-11). If we get back to the Euler equations, translated by 1.3-4-5), one obtains exactly the same equations with $\alpha = 0$.

This immediately suggests to reformulate the Euler equations as the (very singular) limit of the Vlasov-Poisson system when α vanishes.

The new formulation involves a distribution function f(t,x,v), with the same Vlasov equation (1.6) :

(1.13) $\partial_t f + \mathrm{div}_x(vf) + \mathrm{div}_v(Ef) = 0,$

the same field equation (1.7) :

(1.14) $E(t,x) = -\,\mathrm{grad}_x \Phi(t,x),$

and the algebraic equation :

(1.15) $\int f(t,x,v)\, dv = 1$

instead of the Poisson equation (1.8).

It is striking that such a model has already been considered in Plasma Physics (B. Perthame, personal communication), without noticing a link with the Euler equation ! It corresponds to a "quasi neutrality" assumption.

Notice that Φ plays the role of a Lagrange multiplier corresponding to the algebraic constraint (1.15).

Let us now check that the new "kinetic" formulation of the Euler equations is consistent with the classical one :

Propostion 1

• Let $v = V(t,x)$ and $\phi = \phi(t,x)$ be a smooth velocity field and a smooth pressure field that satisfy the Euler equations (1.1-2bis) on the torus $\mathbb{T}^d$. Then :

(1.16) $f(t,x,v) = \delta(v-V(t,x))$

and :

(1.17) $E(t,x) = -\,\mathrm{grad}_x \phi(t,x)$

are solutions to the "Kinetic" formulation (1.13-14-15) of the Euler equations. •

Though the result is almost obvious, let us give a proof.

Proof

The proof follows from straightforward computations. The density equation (1.15) is automatically satisfied by (1.16). Let us check that (1.13) holds in the sense of distributions, that is :

$$(1.18)\qquad 0 = \iiint f(t,x,v)\left[\partial_t Z(t,x,v) + v.\mathrm{grad}_x Z(t,x,v) + E(t,x).\mathrm{grad}_v Z(t,x,v)\right] dx\, dv\, dt$$

for all smooth compactly supported function Z defined on $\mathbb{R} \times \mathbb{T}^d \times \mathbb{R}^d$.

Since V and $E = -\mathrm{grad}_x \phi$ satisfy the classical Euler equations, one has :

$$(1.19)\qquad \partial_t V_i + V_j \partial_{x_j} V_i = E_i \qquad \text{and} \qquad \partial_{x_j} V_j = 0$$

(where the usual implicit sommation convention is used).

Equation (1.18) can be rewritten as :

$$(1.20)\qquad 0 = \iint \left[(\partial_t Z)\,(t,x,V)) + (\partial_{x_j} Z)\,(t,x,V(t,x))\; V_j(t,x) + E_i(t,x)\,(\partial_{v_i} Z)\,(t,x,V(t,x)) \right] dx\, dt$$

by using definition (1.16).

Since :

$$\partial_t [\, Z(t,x,V(t,x))] = (\partial_t Z)\,(t,x,V(t,x)) + (\partial_{v_i} Z)\,(t,x,V(t,x)) \frac{\partial V_i}{\partial t}(t,x)$$
$$= (\text{by (1.19)})\,(\partial_t Z)\,(t,x,V(t,x)) + E_i(t,x)\,(\partial_{v_i} Z)\,(t,x,V(t,x))$$
$$- V_j(t,x)\,(\partial_{x_j} V_i)\,(t,x)\,(\partial_{v_i} Z)\,(t,x,V(t,x))$$

and :

$$\partial_{x_j}[\, V_j\, Z(t,x,V(t,x))] = (\partial_{x_j} Z)\,(t,x,V(t,x))\; V_j(t,x)$$
$$+ \partial_{x_j} V_j(t,x)\; Z(t,x,V(t,x)) + (\partial_{v_i} Z)\,(t,x,V(t,x))\; \partial_{x_j} V_i(t,x)\; V_j(t,x)$$
$$= V_j\,(\partial_{x_j} Z) + V_j\,(\partial_{x_j} V_i)\,(\partial_{v_i} Z) \qquad (\text{since } \partial_{x_j} V_j = 0 \text{ by (1.19)}),$$

one gets that the integrand of (1.20) is an exact derivative :

$$\partial_t [Z(t,x,V(t,x))] + \partial_{x_j}[V_j(t,x)\; Z(t,x,V(t,x))]$$

and, therefore, equality (1.20) holds since Z is compactly supported. ▬

Let us now compare the Kinetic formulation and the concept of mv-solutions introduced by Diperna and Majda [6]. When the velocities are assumed to be bounded (for simplicity), then a mv-solutions is a time and space parameterized probability mesure (with compact support) on the velocity space $\mathbb{R}^d$: $(t,x) \in \mathbb{R} \times \mathbb{T}^d \to f(t,x,dv) \in \mathrm{Prob}(\mathbb{R}^d)$, with suitable measurability assumptions, that satisfies the following equations :

$$(1.21)\qquad \mathrm{div}_x \langle v \rangle = 0,$$

(1.22) $\partial_t < v > + \mathrm{div}_x < v \otimes v > + \mathrm{grad}_x \phi = 0$

in the sense of distributions (for some distribution ϕ which is not necessarily a function), where :

(1.23) $< v > = \int v f(.,.,dv), \quad < v \otimes v > = \int v \otimes v \, f(.,.,dv).$

Equations (1.21)-(1.22) exactly correspond to the two first moment equations that can be derived from the kinetic formulation (1.13-14-15). Indeed, (1.21) follows from (1.13) integrated over $\mathbb{R}^d$ with respect to v and (1.22) follows from (1.13), multiplied by v and then integrated in v. In both cases, (1.14) and (1.15) must be taken into account.

Thus, an m-v-solution can be seen as an "incomplete" kinetic solution that satisfies only two moment equations instead of the complete system (1.13-14-15).

Let us now go back to the kinetic formulation (1.13-14-15). The analysis of this system seems very hard. Indeed, the only natural (formal) estimate involves the "kinetic energy" :

(1.24) $$\frac{d}{dt} \frac{1}{2} \iint |v|^2 f(t,x,v) \, dv \, dx = 0,$$

and there is no a priori bound on the force field. In comparison, for the Vlasov-Poisson system, the energy conservation is given by :

(1.25) $$\frac{d}{dt} \left\{ \frac{1}{2} \iint |v|^2 f(t,x,v) \, dv \, dx + \frac{\alpha}{2} \int |E(t,x)|^2 \, dx \right\} = 0$$

and there is an a priori bound, when $\alpha > 0$ is fixed, on the L^2 norm of E. Thanks to the new tools recently developped for the transport equations [8,...], (1.25) is sufficient to guarantee a global existence theorem for the Vlasov-Poisson system, as shown by Diperna and Lions [5].

Unfortunately, when $\alpha \to 0$, there is no uniform bound on the force field in L^2 and the limiting process seems very delicate.

The pressure field formally satisfies a Laplace equation :

$$- \Delta_x \phi(t,x) = D_x^2 : \int v \otimes v \, f(t,x,v) \, dv,$$

that can be easily deduced from (1.13-14-15). This property, together with (1.24), almost gives an L^1 control on ϕ, which is far from being sufficient to give a sense to (1.13), where the multiplication of $E = - \mathrm{grad}_x \phi$ by f is involved.

So, it turns out that a global existence theorem for the Kinetic formulation (1.13-14-15) of the Euler equations is an entirely open problem.

Some additional informations can be obtained in the one-dimensional case (notice that the kinetic formulation can lead to non trivial solutions in the one-dimensional case -as shown in section 4- as opposed to the classical formulation of the Euler equation). Indeed, by setting :

$$(1.26) \qquad m_k(t,x) = \int v^k f(t,x,v)\, dv, \qquad k = 0,1,2,3,\ldots,$$

it can be easily deduced from (1.13-14-15) that :

$$(1.27) \qquad m_0 \equiv 1, m_1 \equiv 0 \ ^{(*)} \text{ and :}$$

$$\partial_t m_k + \partial_x m_{k+1} = k\, E m_{k-1}, \quad \text{for } k = 2,3,\ldots$$

Moreover E is simply given by :

$$E = \partial_x m_2.$$

Thus, the following infinite system is obtained :

$$(1.28) \qquad \partial_t m_k + \partial_x m_{k+1} = k\, \partial_x m_2 . m_{k-1}, \quad k = 2,3,\ldots$$

This system has nice algebraic properties. Indeed, by using polynomial changes of unknowns, it seems possible to get an infinite system of conversation laws :

$$(1.29) \qquad \partial_t H_k + \partial_x Z_{k+1} = 0, \quad k = 2,3,\ldots$$

where H_k, Z_k are polynomial functions of $m_2,\ldots,m_k$.

(*) As a matter of fact, one gets $m_1(t,x) = \text{cst}$. However, because of the Galilean invariance of the equation, $m_1 = 0$ is a suitable normalization.

We have not been able to compute them so far, but the first conversation laws are :

$$(1.30)\qquad \begin{cases} \partial_t m_2 + \partial_x m_3 = 0 \\ \partial_t m_3 + \partial_x (m_4 - \frac{3}{2} m_2^2) = 0 \\ \partial_t (m_4 - 2m_2^2) + \partial_x (m_5 - 4m_2 m_3) = 0 \\ \text{etc} \dots \end{cases}$$

In particular, one gets the following conservation properties :

$$(1.31)\qquad \frac{d}{dt}\iint v^3 f(t,x,v)\, dv\, dx = 0, \frac{d}{dt}\left\{ \iint v^4 f(t,x,v)\, dv\, dx - 2\int \left(\int v^2 f(t,x,v)\, dv \right)^2 dx \right\} = 0$$

etc...

This might indicate a strong algebraic structure behind (1.28).

II - Sequences of solutions to the Euler equations with a uniform control on the pressure field

Let us consider a family $(v^\varepsilon(t,x), \phi^\varepsilon(t,x))$ of solutions to the Euler equations on the d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ ($d = 2$ or 3).

$$(2.1)\qquad \left\{ \begin{array}{l} \left. \begin{array}{l} \operatorname{div}_x v^\varepsilon = 0 \\ \partial_t v^\varepsilon + \operatorname{div}_x (v^\varepsilon \otimes v^\varepsilon) = E^\varepsilon \\ E^\varepsilon(t,x) = -\operatorname{grad}_x \phi^\varepsilon \end{array} \right\} t \in \mathbb{R}, x \in \mathbb{T}^d \\ v^\varepsilon(0,x) = v_0^\varepsilon(x), \ x \in \mathbb{T}^d \end{array} \right.$$

The goal of this section is to describe the behaviour, when $\varepsilon \to 0$, of such a family when there is a uniform control on the pressure field ϕ^ε and on the corresponding force $E^\varepsilon = -\operatorname{grad}_x \phi^\varepsilon$. Typically, it will be assumed that :

$$(2.2)\qquad \|\partial_t E^\varepsilon\| + \|D_x E^\varepsilon\| \le C \text{ uniformly on } t,x \text{ and } \varepsilon.$$

Moreover, to avoid some technical difficulties,

$$(2.3)\qquad \|v_0^\varepsilon(x)\| \le R, \text{ uniformly on } x \text{ and } \varepsilon,$$

will also be assumed (although a more natural assumption would have been :

$$(2.4) \qquad \int |v_0^\varepsilon(x)|^2 \, dx \leq C).$$

Clearly, there is no serious reason to assume (2.2) since there is no a priori estimate of this type in the Euler equations, ! Nevertheless, there are interesting examples (that will be discussed later) where v^ε is more and more oscillating when ε goes to 0 and (2.2) holds.

In such conditions, the behaviour of $(v^\varepsilon, E^\varepsilon)$ can be described by :

Proposition 2

Let $(v^\varepsilon, \phi^\varepsilon)$ a sequence of smooth solutions to the Euler equations on the torus $\mathbb{T}^d$ that satisfies (2.2-3). Then, for a subsequence, still labelled by ε, there is a time parameterized probability measure on the phase space : $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$, $t \to f(t,dx,dv)$, compactly supported, and a Lipschitz continuous potential force field :

$$(2.5) \qquad E(t,x) = - \operatorname{grad}_x \phi(t,x),$$

that are solutions to the "kinetic formulation" (1.13-14-15) of the Euler equations and such that :

$$(2.6) \qquad \iint Z(x,v) \, f(t,dx,dv) = \lim_\varepsilon \int Z(x,v^\varepsilon(t,x)) \, dx$$

for any $t \in \mathbb{R}$ and any continuous function Z on $\mathbb{T}^d \times \mathbb{R}^d$. ∎

In addition, f can be explicitely computed by using the integral curves :

$$t \to (X(t,x,v),W(t,x,v))$$

of the Hamiltonian ordinary differential system in $\mathbb{T}^d \times \mathbb{R}^d$:

$$(2.7) \qquad \dot{x} = v, \ \dot{v} = E(t,x),$$

that satisfy the initial conditions :

$$(2.8) \qquad X(0,x,v) = x, \ W(0,x,v) = v \quad \text{for any } x,v.$$

Indeed, one has :

Proposition 3

In proposition 2, f can be defined, at any time t, by :

$$(2.9) \qquad \iint Z(x,v)\, f(t,dx,dv) = \iint Z(X(t,x,v),W(t,x,v))\, f_0(x,dv)\, dx$$

For any compactly supported continuous function Z on $\mathbb{T}^d \times \mathbb{R}^d$, where f_0 is the "initial distribution function" defined by :

$$(2.10) \qquad \iint Z(x,v)\, f_0(x,dv) = \lim_{\varepsilon} \int Z(x,v_0^{\varepsilon}(x))\, dx \quad \blacksquare$$

Remark

Behind this kinetic description of the limit when ε, one recovers a very precise Lagrangian picture of the motion. Each "material" particle follows, in the physical space, the curve $t \to X(t,x,v) \in \mathbb{T}^d$, where the initial velocity v is distributed in $\mathbb{R}^d$ according to the probability law $f_0(x,dv)$, and x is the initial position at time 0 in the physical space. Thus the flowmap of the particles, in the physical space, is in general not deterministic (from a given point x at time 0, different trajectories can be followed), in contradiction with the classical axioms of continuum mechanics. As a matter of fact, this "generalized flowmap" (in the sense of [3]) is the projection onto the physical space of the integral curves of the Hamiltonian system $\dot{x} = v$, $\dot{v} = E(t,x)$ that lies in the phase space $\mathbb{T}^d \times \mathbb{R}^d$ above the physical space $\mathbb{T}^d$.

In [3], a "generalized" flow has been defined as a probability measure $q(d\xi)$ on the set $\Omega = (\mathbb{T}^d)^{\mathbb{R}}$ of all paths $\mathbb{R} \ni t \to \xi(t) \in \mathbb{T}^d$, valued in the physical space. Here, the "generalized" flow $q(d\xi)$ associated with the Kinetic solution f(t,dx,dv) can be uniquely defined by :

$$(2.11) \qquad \int_{\Omega} Z(\xi(t_1),\ldots,\xi(t_n))\, q(d\xi) = \int_{\mathbb{T}^d \times \mathbb{R}^d} \int Z(X(t_1,x,v),\ldots,X(t_n,x,v)) f_0(x,dv)\, dx,$$

for any finite sequence $t_1,\ldots,t_n$ in $\mathbb{R}$ and any continuous function Z on $(\mathbb{T}^d)^n$.

This can be equivalently expressed by saying that :

i) q-almost surely, a path $\xi(t)$ is the x-component of an integral curve $(X(t,x,v),V(t,x,v))$ of the Hamiltonian system $\dot{x} = v$, $\dot{v} = E(t,x)$;

ii) at time 0, $x = \xi(0)$ and $v = \dot{\xi}(0)$ are correlated by the probability law f(x,dv).

Moreover, it follows from a result of [3] that such a flow satisfies a generalized version of the least action principle. ▬

Proof of Propositions 2 and 3

Since v^ε, ϕ^ε are smooth solutions to the Euler equations, each material particle follows a trajectory $t \to \xi(t)$ that satisfies the dynamical equation :

$$\ddot{\xi} = E^\varepsilon(t,\xi)$$

Thus, if we denote by $g^\varepsilon(t,x) \in \mathbb{T}^d$ the position at time t of the particle issued from x at time 0, one has :

$$\frac{\partial^2 g^\varepsilon}{\partial t^2}(t,x) = E^\varepsilon(t,g^\varepsilon(t,x))$$

$$g^\varepsilon(0,x) = x, \frac{\partial g^\varepsilon}{\partial t}(0,x) = v_0^\varepsilon(x).$$

Then, it is natural to consider, in the phase space $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$, the Hamiltonian system :

$$\dot{x} = v\ ,\ \dot{v} = E^\varepsilon(t,x),\ \text{(where } E^\varepsilon = -\ \text{grad}_x\, \phi^\varepsilon),$$

and the corresponding integral curves :

$$(t,x,v) \in \mathbb{R} \times \mathbb{T}^d \times \mathbb{R}^d \to (X^\varepsilon(t,x,v),\ W^\varepsilon(t,x,v)) \in \mathbb{T}^d \times \mathbb{R}^d$$

that satisfy :

$$X^\varepsilon(0,x,v) = x, \quad W^\varepsilon(0,x,v) = v.$$

The "flowmap" g^ε can now be easily deduced from $(X^\varepsilon, W^\varepsilon)$ and the initial velocity $v^\varepsilon{}_0$ by :

$$(2.12) \qquad g^\varepsilon(t,x) = X^\varepsilon(t,x,v_0^\varepsilon(x)),\ \frac{\partial g^\varepsilon}{\partial t}(t,x) = W^\varepsilon(t,x,v_0^\varepsilon(x))$$

Notice that, because of the incompressibility condition,

(2.13) $x \to g^\varepsilon(t,x)$ is a volume preserving diffeomorphism from $\mathbb{T}^d$ into itself, and that the velocity field $v^\varepsilon(t,x)$ is given by :

$$(2.14) \qquad v^{\varepsilon}(t,g^{\varepsilon}(t,x)) = \frac{\partial g^{\varepsilon}}{\partial t}(t,x), \quad \text{for any } t \in \mathbb{R}, x \in \mathbb{T}^d.$$
$$= W^{\varepsilon}(t,x,v_0^{\varepsilon}(x))$$

To prove propositions 2 and 3, a key step is to remark that, due to the uniform Lipschitz continuity of E^{ε}, the integral curves $(X^{\varepsilon},W^{\varepsilon})$ behave nicely when ε goes to 0, even if the velocity field $v^{\varepsilon}(t,x)$ and the "flowmap" $g^{\varepsilon}(t,x)$ get more and more oscillating.

Indeed, from Ascoli's theorem and the elementary theory of ODE's, it follows that, for a subsequence still labelled by ε, there is a globally Lipschitz continuous force field $E(t,x) = -\text{grad}_x \phi(t,x)$ such that the integral curves $(X^{\varepsilon},W^{\varepsilon})$ uniformly converge on any compact subset of $\mathbb{R} \times \mathbb{T}^d \times \mathbb{R}^d$ toward the integral curves (X,W) corresponding to the Hamiltonian system $\dot{x} = v$, $\dot{v} = E(t,x)$.

Remark

According to the new theory on ODE's by Diperna and Lions, an integral Lipschitz uniform bound on E^{ε} (say, a bound in the Sobolev space $W^{1,1}$) is enough to preserve the strong convergence of the integral curves $(X^{\varepsilon},W^{\varepsilon})$ (but in a L^1 sense with respect to x and v). Then, the results of Propositions 2 and 3 may stay valid under weaker assumptions. However, the particles are located, at time 0 and for ε fixed, in the phase space (x,v), on a negligible set $\{(x,v) ; v = V^{\varepsilon}_0(x)\}$, and it is therefore unclear how Diperna and Lions theory can be used.

Let us now consider the behaviour of the initial velocity field V^{ε}_0 when ε goes to 0. A uniform bound has been assumed (2.3) : $\| v^{\varepsilon}_0(x) \| \leq R$. It is therefore easy to describe the behaviour of v^{ε}_0 in terms of Young's measure, following [9,7,1...]. For each ε, one can consider the x-dependent probability measure on the ball $B = \{v \in \mathbb{R}^d ; \| v \| \leq R\}$ given by :

$$f_0^{\varepsilon}(x ; dv) = \delta(v - v_0^{\varepsilon}(x)), \quad x \in \mathbb{T}^d.$$

The sequence (f^{ε}_0) is weak-* compact in the dual space $L^{\infty}(\mathbb{T}^d ; C(B)')$ and has a cluster point f_0.

Thus, for a subsequence still labelled by ε, there is a x-dependent probability measure on B, $f_0(x ; dv)$ such that :

$$(2.15) \qquad \int Z(x,v_0^{\varepsilon}(x))\, dx = \iint Z(x,v)\, f_0^{\varepsilon}(x ; dv)\, dx \to \iint Z(x,v)\, f_0(x ; dv)\, dx,$$

for all continuous function Z on $\mathbb{T}^d \times B$ (see [1] for more details).

Remark

Assumption (2.3) could have been replaced by the more natural hypothesis :

$$(2.16) \qquad \int \| v_0^\varepsilon(x) \|^2 \, dx \leq C.$$

Then, the behaviour of (v^ε_0) would have been described in terms of "generalized" Young's measures as in [6]. Here (2.3) is assumed to make the proof simpler.

By combining the uniform convergence of the integral curves $(X^\varepsilon, W^\varepsilon)$ toward (X,W) on any compact subset of $\mathbb{R} \times \mathbb{T}^d \times \mathbb{R}^d$ and the weak-* convergence of f^ε_0 toward f_0, one easily deduces that :

$$(2.17) \qquad \int Z(X^\varepsilon(t,x,v_0^\varepsilon(x)), W^\varepsilon(t,x,v_0^\varepsilon(x))) \, dx \to \iint Z(X(t,x,v), W(t,x,v)) \, f_0(x,dv) \, dx,$$

for any continuous function Z continuous on $\mathbb{T}^d \times \mathbb{R}^d$ and any time $t \in \mathbb{R}$.

The right hand side of (2.17) defines a probability measure, with compact support, on $\mathbb{T}^d \times \mathbb{R}^d$, for any time t, by setting :

$$(2.18) \qquad \iint Z(x,v) \, f(t,dx,dv) = \iint Z(X(t,x,v), W(t,x,v)) \, f_0(x,dv) \, dx$$

(the fact that f(t,.,.) is compactly supported follows from the fact that i) f_0 is compactly supported (in $\mathbb{T}^d \times B$), ii) E(t,x) is globally Lipschitz continuous and therefore :

$$\| X(t,x,v) - x \| + \| W(t,x,v) - v \| \leq C \, | t |$$

for some constant C).

By definition (2.18) and by (2.17), one has :

$$(2.19) \qquad \iint Z(x,v) \, f(t,dx,dv) = \lim_\varepsilon \int Z(X^\varepsilon(t,x,v_0^\varepsilon(x)), W^\varepsilon(t,x,v_0^\varepsilon(x))) \, dx$$

$$= (\text{by (2.12-14)}) \lim_\varepsilon \int Z(g^\varepsilon(t,x), V^\varepsilon(t,g^\varepsilon(t,x)) \, dx$$

$$= \lim_\varepsilon \int Z(x,V^\varepsilon(t,x)) \, dx \quad (\text{since } x \to g^\varepsilon(t,x) \text{ is volume preserving by (2.13)})$$

Thus the first part of Proposition 2 is proved, as well as Proposition 3 directly follows from (2.18).

It is now enough to check that f and E are "kinetic" solutions to the Euler equations. Once again, the proof is easy. Let us consider a smooth compactly supported function Z on $\mathbb{T}^d \times \mathbb{R}^d$. By (2.18), one has :

$$\frac{d}{dt}\iint Z(x,v)\, f(t,dx,dv) =$$

$$\iint [(\mathrm{grad}_x Z)\,(X(t,x,v), W(t,x,v))\, \partial_t X(t,x,v)$$

$$+ (\mathrm{grad}_v Z)\,(X(t,x,v), W(t,x,v))\, \partial_t W(t,x,v)]\, f_0(x,dv)\, dx$$

$$= \text{(by definition of the integral curves X,W)}$$

$$\iint [(\mathrm{grad}_x Z)\,(X,W).W + (\mathrm{grad}_v Z)\,(X,W).E(t,X)]\, f_0(x,dv)\, dx$$

$$= \text{(again by definition (2.18))} :$$

$$\iint [(\mathrm{grad}_x Z)\,(x,v).v + (\mathrm{grad}_v Z)\,(x,v).E(t,x)]\, f(t,dx,dv)$$

This proves that f satisfies (1.13) in the sense of distributions and achieves the proof of propositions 2 and 3. ∎

Remark

By using the same technique, on deduces also that any sequence of "kinetic" solutions $(f^\varepsilon,\phi^\varepsilon)$ to the Euler equations, for which the uniform bounds :

(2.20) $\quad \mathrm{supp}\,(f_0^\varepsilon) \subset \mathbb{T}^d \times$ Ball of radius R, R independent of ε,

(2.21) $\quad \| D^2_{t,x} \phi^\varepsilon(t,x) \| \leq C,$

hold, has a subsequence converging toward a kinetic solution.

III - Two examples of sequences of classical solutions

Two explicit examples of sequences of smooth solutions $(v^\varepsilon,\phi^\varepsilon)$ of the Euler equations, showing the behaviour predicted by the results of section II (Propositions 2 and 3), are discussed in this section. Both are time-independent (and, therefore, $v^\varepsilon(t,x) = v^\varepsilon_0(x)$), one is two-dimensional ($x \in \mathbb{T}^2$), and the other one is three-dimensional and slightly out of the frame

discussed in section two (the physical space is $\mathbb{R}^2 \times \mathbb{T}^1$ instead of the torus $\mathbb{T}^3$; nevertheless, the velocity field is fastly decaying in two directions and periodical with respect to the other one).

a) the 2-dimensional case

The formulas are as follows :

$$x = (x_1, x_2) \in \mathbb{T}^2,$$

(3.1) $$v^\varepsilon = v^\varepsilon(x) = \mathrm{rot}\, \psi^\varepsilon(x) \quad \text{where } \psi^\varepsilon(x) = \frac{\varepsilon}{\pi} \sin(\pi x_1) \sin(\pi \frac{x_2}{\varepsilon}),$$

where $\varepsilon^{-1} = 1,2,3,\ldots,$

(3.2) $$\phi^\varepsilon = \phi^\varepsilon(x) = \frac{1}{4\pi} \left\{ \sin(2\pi x_1) + \varepsilon^2 \sin(2\pi \frac{x_2}{\varepsilon}) \right\},$$

(3.3) $$v^\varepsilon(x) = \left(\sin(\pi x_1) \cos(\pi \frac{x_2}{\varepsilon}) , - \varepsilon \cos(\pi x_1) \sin(\pi \frac{x_2}{\varepsilon}) \right) = v_0^\varepsilon(x)$$

As advertised, $(v^\varepsilon, \phi^\varepsilon)$ solve the Euler equations and bounds (2.2-3) are satisfied :

(3.4) $$\| v_0^\varepsilon(x) \|^2 = \| v^\varepsilon(x) \|^2 \le 2, \quad | \partial_{x_i x_j} \phi^\varepsilon(x) | \le \pi$$

Then, straightforward calculations show that, when ε goes to zero, the force field is given by :

(3.5) $$E(x) = (- \frac{1}{2} \cos 2\pi x_1, 0) = - \phi'(x_1), \quad \phi(x_1) = \frac{1}{4\pi} \sin(2\pi x_1),$$

and the distribution function $f(t,dx,dv) = f_0(x,v)\, dxdv$ is implicetely defined by :

(3.6) $$\int_0^1 \int_0^1 \int_0^1 \int_0^1 Z(x_1, x_2, v_1, v_2)\, f(t,dx,dv)$$
$$= \int_0^1 \int_0^1 \int_0^1 Z(x_1, x_2, \sin(\pi x_1) \cos(\pi\theta), 0)\, dx_1, dx_2\, d\theta,$$

for any continuous function Z on $\mathbb{T}^2 \times \mathbb{R}^2$, or, more explicetely, by :

$$(3.7)\qquad f(t,dx,dv) = f_0(x,v)dxdv,\ f_0(x,v) = \begin{cases} \frac{1}{\pi}(\sin^2(\pi x_1)-v_1^2)^{-\frac{1}{2}}\,\delta(v_2) \\ \text{if } |v_1| \le |\sin \pi x_1| \\ 0 \text{ otherwise} \end{cases}$$

Remark

The "generalized flowmap" (cf.[3] and the remark made right after Proposition 2) associated with this kinetic solution is the probability measure q on the set Ω of all paths $t \in \mathbb{R} \to \xi(t) \in \mathbb{T}^2$ such that :

i) q-almost surely, ξ is of the form :

$$\xi(t) = (\xi_1(t), \xi_2(0)) \text{ where } \xi_1 \text{ solves } \ddot{\xi}_1 = -\frac{1}{2}\cos 2\pi\xi_1$$

(the pendulum equation) and satisfies the initial condition.

ii) $\dot{\xi}_1(0) = \sin(\pi\xi_1(0))\cos(\pi\theta)$ where θ is uniformly distributed in [0,1].

Typical trajectories are shown on figure 1. Notice that, when $\varepsilon \to 0$, one space dimension "disappears" and this "dimensional collapse" randomizes the flow.

b) the 3-dimensional case

For $x = (x_1,x_2,x_3) \in \mathbb{R}^2 \times \mathbb{T}^1$, we set, for each fixed $\varepsilon = 1/\pi, 2/\pi, 3/\pi, \ldots$

$$(3.8)\qquad v^\varepsilon = v^\varepsilon(x) = v_0^\varepsilon(x) = e^{-\frac{1}{2}(x_1^2+x_2^2)}\left(\cos\frac{x_3}{\varepsilon}, \sin\frac{x_3}{\varepsilon}, \varepsilon x_1 \sin\frac{x_3}{\varepsilon} - \varepsilon x_2 \cos\frac{x_3}{\varepsilon}\right)$$

and

$$(3.9)\qquad \phi^\varepsilon = \phi(x) = \frac{1}{2}e^{-(x_1^2+x_2^2)}$$

It can be easily checked that $(v^\varepsilon,\phi^\varepsilon)$ are stationnary solutions to the Euler equations. It is remarkable that the pressure does not depend on ε ! As for the former example, the velocity field is more and more oscillating when $\varepsilon \to 0$ (notice that the third component goes strongly to zero).

This case does not exactly fit in the framework of section 2. However, because of the x_3-periodicity and the fast decay of $v^\varepsilon(x_1,x_2,x_3)$ when (x_1,x_2) goes to infinity, the same kind of analysis can be carried out.

The distribution function obtained when $\varepsilon \to 0$ is implicitely given by :

$$(3.10)\qquad \iiint\iiint Z(x_1,x_2,x_3,v_1,v_2,v_3)\, f(t,dx,dv)$$

$$= \iiint\iint_0^1 Z(x_1,x_2,x_3,e^{-\frac{1}{2}(x_1^2+x_2^2)}\cos\pi\theta, e^{-\frac{1}{2}(x_1^2+x_2^2)}\sin\pi\theta, 0)\, dx_1\, dx_2\, dx_3\, d\theta$$

or, more explicitely, by :

$$(3.11)\qquad f(t,dx,dv) = f_0(x,dv)dx,\ f_0(x,dv) = \delta(v_3)\,\delta(v_1^2+v_2^2 - e^{-(x_1^2+x_2^2)})\,\pi^{-1}.$$

Remark

The corresponding Lagrangian description is provided by the generalized flowmap q which is the probability measure q on the set Ω of all paths $\mathbb{R} \ni t \to \xi(t) \in \mathbb{R}^2 \times \mathbb{T}^1$ defined by :

q-almost surely $\xi(t)$ is of the form :

$$\xi(t) = (\xi_1(t), \xi_2(t), \xi_3(0))$$

where (ξ_1, ξ_3) solves the differential equation :

$$(\ddot{\xi}_1,\ddot{\xi}_2) = e^{-\frac{1}{2}(\xi_1^2+\xi_2^2)}(\xi_1,\xi_2),$$

and the initial conditions :

$$(\dot{\xi}_1(0), \dot{\xi}_2(0)) = e^{-\frac{1}{2}(\xi_1^2(0)+\xi_2^2(0))}(\cos\theta, \sin\theta)$$

where θ is uniformly distributed in $[0,\pi]$.

IV - A family of "Kinetic" solutions to the Euler equations

This family has already been introduced in the framework of generalized flows [3].

These solutions are time independent and the distribution function is of the form :

$$(4.1)\qquad f(t,dx,dv) = f_0(x,dv)\, dx = \lambda\,(\frac{\|v\|^2}{2} + \phi(x))\, dx\, dv,$$

where λ is a non negative function, while the force field is given by :

(4.2) $\quad E = E(x) = -\,\mathrm{grad}\phi(x),$

and ϕ is any given smooth function defined on $\mathbb{T}^d$.

It is easy to check that :

(4.3) $\quad \mathrm{div}_x(vf) + \mathrm{div}_v(Ef) = 0$

holds in the sense of distribution for any choice of λ.

Thus, f will be a kinetic solution to the Euler equations if and only if (1.15) holds, i.e. :

(4.4) $$\int \lambda \left(\frac{\|v\|^2}{2} + \phi(x)\right) dv \equiv 1, \quad \lambda \geq 0,$$

or, equivalently :

(4.5) $$\int \lambda \left(\frac{\|v\|^2}{2} + c\right) dv = 1, \quad \forall c \in K, \quad \lambda \geq 0,$$

where K is the interval $K = [\min\phi, \max\phi]$.

By using polar coordinates, (4.5) can be rewritten as :

(4.6) $$\alpha_d \int_0^{\infty} \lambda(s+c) s^{\frac{d}{2}-1} ds = 1, \quad \forall c \in K, \ \lambda \geq 0,$$

where α_d depends on the space dimension.

Except the trivial case when ϕ is a constant ϕ_0 (then any choice of λ such that $\int \lambda[(\|v\|^2/2)+\phi_0]dv = 1$ is convenient), this integral equation leads to :

Proposition 4

Let ϕ be a smooth non constant function given on $\mathbb{T}^d$. Let us denote by $\mathrm{Max}\phi$ the maximum of ϕ and define a stationnary distribution function f(x,v) by (4.1).

Then, (f,ϕ) is a Kinetic solution to the Euler equations (i.e. satisfies (1.13-14-15)) :

(4.7) i) iff $\lambda(s) = \begin{cases} \frac{1}{\pi\sqrt{2}}(\max\phi - s)^{-\frac{1}{2}} & \text{if } s \le \max\phi \\ 0 \quad \text{otherwise} \end{cases}$, when $d = 1$;

(4.8) ii) iff $\lambda(s) = \delta(s - \max\phi)$, when $d = 2$;

iii) in no case when $d = 3$.

Indeed, equation (4.6) has no non negative solution λ, when $d = 3$, and (4.7), (4.8) defines the unique solution for $d = 1$ and $d = 2$.

Remark 1

The explicit solutions obtained in section 3 are precisely of this form (compare (4.1-7) and (3.7), (4.1-8) and (3.11)).

Remark 2

It is remarkable that the distribution functions f of form (4.1) are m.v. solutions to the Euler equations in the sense of Diperna and Majda, without any restriction on $\lambda \ge 0$ (in other words they automatically satisfy the 2 first moment equations of the kinetic formulation, as explained in section 1).

V - Numerical simulations

A numerical scheme has been introduced in [4] to solve the Euler equations. Although this particle method has been designed for classical solutions, there is no difficulty to use it for generalized kinetic solutions. This is a major difference with most of the existing schemes designed to solve the Euler equations (finite differences, spectral methods or vortex methods).

Let us briefly review the main aspects of our scheme :

1) The physical domain is divided into N cells of equal volume ; the cells are denoted by C_i, $i=1,...,N$ and their center of mass by x_i ;

2) At time 0, we are given one particle per cell C_i with position $X^0_i = x_i$ and given velocity V^0_i ;

3) At time Δt (where Δt denotes the time step), one denotes by $\tilde{X}^1_i$ the position that the i^{th} particle would occupy if there were no pressure forces, namely : $\tilde{X}^1_i = X^0_i + \Delta t\, V^0_i$;

4) Then, in order to keep constant the density, each cell must be occupied by one particle ; thus a permutation $h = (h(1),\ldots,h(N))$ must be chosen to define the new position $X_i^1 = x_{h(i)}$ of each particle. This permutation is obtained by minimizing the least square criterion :

$$J^1(h) = \sum_{i=1}^{N} \| \tilde{X}_i^1 - x_{h(i)} \|^2$$

5) The new velocity of each particle is then given by :

$$V_i^1 = \frac{1}{\Delta t}(X_i^1 - X_i^0)$$

6) The passage from time step n to time step n+1 is performed in the same way as from step 0 to step 1.

This scheme can be seen as a Lagrangian version of the projection method introduced by Chorin for the incompressible Navier-Stokes equation.

The main step of our scheme is the combinatorial optimization problem that has to be performed at each time step n : Find a permutation h that minimizes :

$$J^n(h) = \sum_{i=1}^{N} C_{ih(i)}^n \quad \text{where} \quad C_{ij}^n = \| \tilde{X}_i^n - x_j \|^2$$

where the $\tilde{X}^n_i$ are deduced from the previous time step. This optimization problem is a classical assignment problem. With the best general methods, the computational cost is of order $O(N^2 \text{ LogN})$ to solve each optimization problem [4]. Due to the very special structure of the "cost matrix" C^n_{ij}, we believe that a more specific algorithm would lead to a computational cost of order 0(N LogN). This is actually true for the one-dimensional case, where the optimization problem is equivalent to a sorting problem. Thus, at the moment, our scheme is efficient only in the one-dimensional case and relevant numerical results have been obtained only in this case.

Fortunately, although there is no non trivial one-dimensional classical solutions to the incompressible Euler equations, the kinetic formulation of the Euler equations leads to non trivial generalized solutions in the one-dimensional case, that can be obtained, through some homogenization process, from classical two-dimensional classical solutions, as discussed in section III. Thus, we can try to get non trivial kinetic solutions, as the ones considered in section IV, by using our numerical scheme. For instance, let us consider the stationnary solutions (4.1) :

(5.1) $$f(t,dx,dv) = f_0(x,dv)\,dx = \lambda\,(\frac{\|v\|^2}{2} + \phi(x))\,dx\,dv$$

where λ is given by (4.7) and ϕ is a given smooth periodic function given on $\mathbb{T}^1 = \mathbb{R}^1/\mathbb{Z}^1$.

This corresponds to a "random velocity field" :

(5.2) $v = r(x)\cos\theta$, where θ is uniformly distributed in $[0,\pi]$ and

$$r(x) = \sqrt{\max\phi - \phi(x)} \times \sqrt{2}$$

Thus, in our code, the initial position and velocity of the i^{th} particle are given by :

(5.3) $$X_i^0 = x_i,\quad V_i^0 = r(x_i)\cos(\frac{x_i}{\varepsilon}),\quad x_i = \frac{i}{N},\quad i = 1,\ldots,N$$

where $\varepsilon > 0$ is a small parameter.

So, the initial velocities are deterministic, but, due to the small parameter ε, they simulate the random velocity field (5.2) and, therefore, one gets a reasonable approximation of the initial distribution function f_0 given by (5.1). In practice, ε has been defined by $\varepsilon = N^{-1/2}$.

Two examples have been considered :

(5.4) $$\phi(x) = \frac{1}{4\pi}\sin(2\pi x)$$

(5.5) $$\phi(x) = \frac{1}{2}(1-x)x$$

Strictly speaking, the second case is out of the framework of this paper since ϕ is not smooth on $\mathbb{T}^1$ (ϕ is only continuous). As a matter of fact, it is more appropriate to consider the problem corresponding to (5.5) as a boundary value problem set on the unit interval with slip boundary condition at x=0 and x=1.

Example (5.5)

On figure 2, some of the particle paths are drawn for the exact solution and two different numerical simulations, both with 512 particles, one with 8 time steps and the second one with 32 time steps. Notice that the scheme becomes unstable when too many time steps are used : for a given time scale and a given time step, a minimum number of particle must be used to preserve a reasonable accuracy for the computed trajectories.

Example (5.4)

On figure 3, the same kind of comparison is shown for (5.4). Notice that the scheme seems considerably less accurate here than for the case (5.5).

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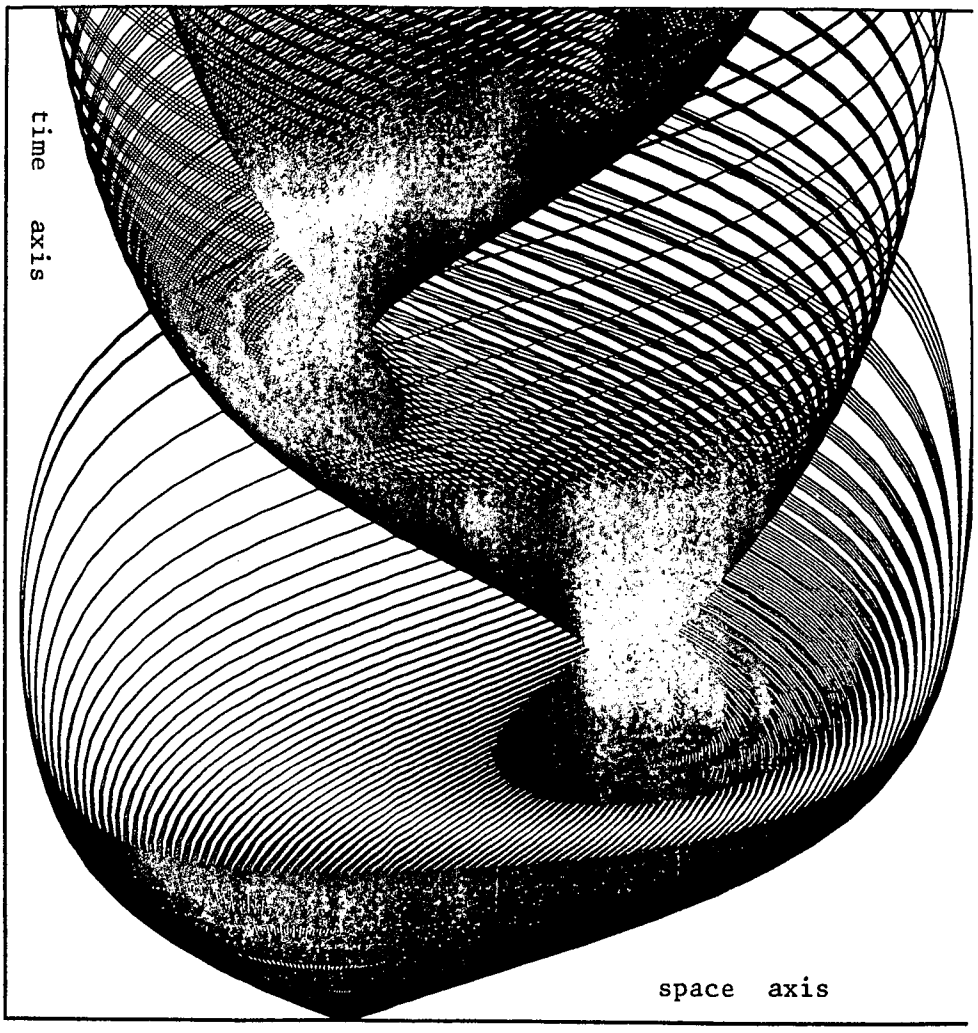

Figure 1. The Lagrangian picture of a kinetic solution of the Euler equations :
A material particle can follow different trajectories $t \to (\xi_1(t),\xi_2(t))$ from a given point $x = (\xi_1(0),\xi_2(0))$.
Here, for a fixed value of x, all solutions of

$$\ddot{\xi}_1 = -\frac{1}{2}\cos(2\pi\xi_1),\ \xi_1(0) = x_1,\ \dot{\xi}_1(0) = \sin(\pi x_1)\cos\theta$$

can be taken, with a uniform probability when θ describes $[0,\pi]$, by the particle initially located at x.

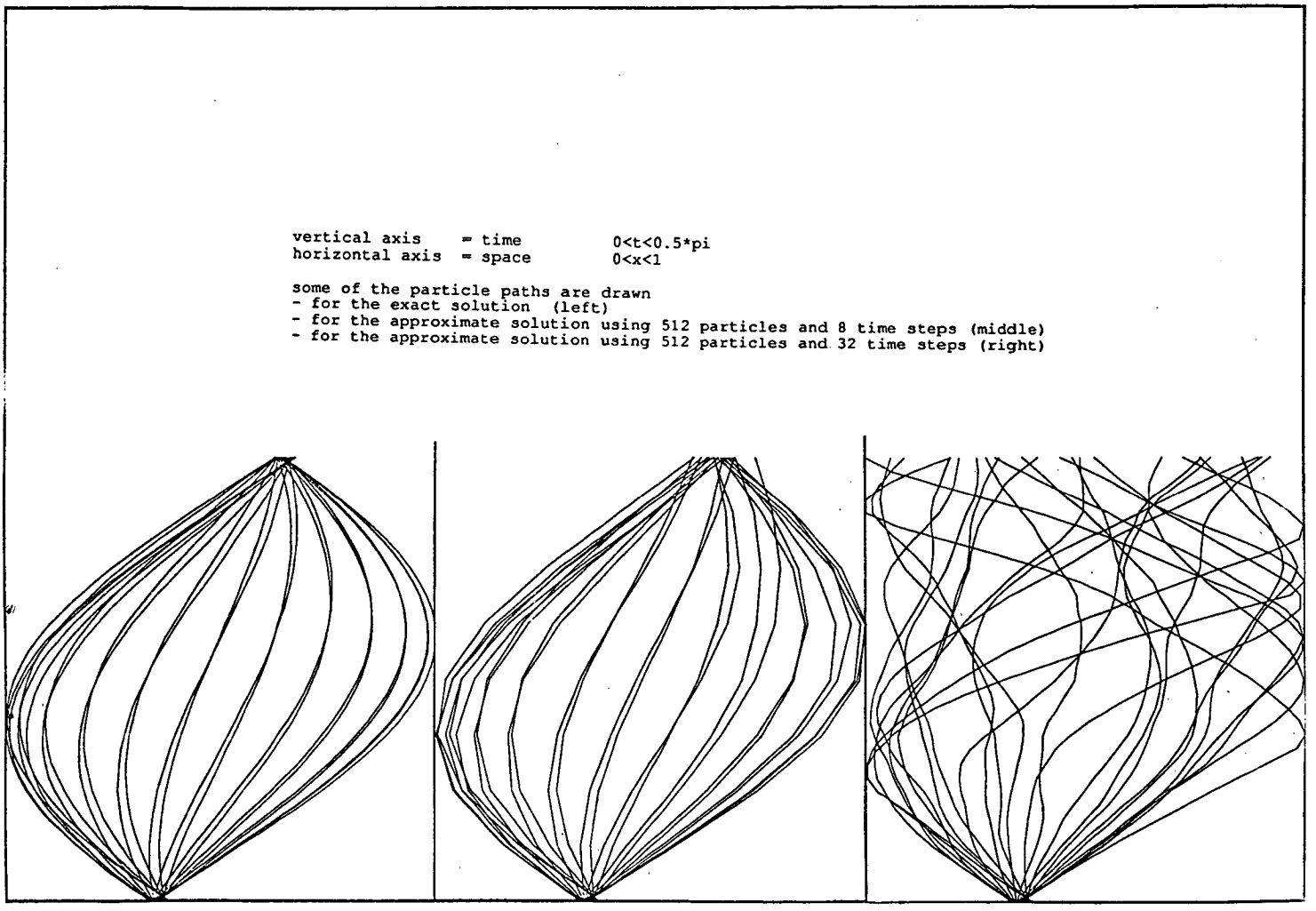
vertical axis = time 0<t<0.5*pi
horizontal axis = space 0<x<1
some of the particle paths are drawn
- for the exact solution (left)
- for the approximate solution using 512 particles and 8 time steps (middle)
- for the approximate solution using 512 particles and 32 time steps (right)

Figure 2.

vertical axis = time 0<t<10/pi
horizontal axis = space 0<x<1

some of the particle paths are drawn
- for the exact solution (left)
- for the approximate solution using 512 particles and 20 time steps (right)

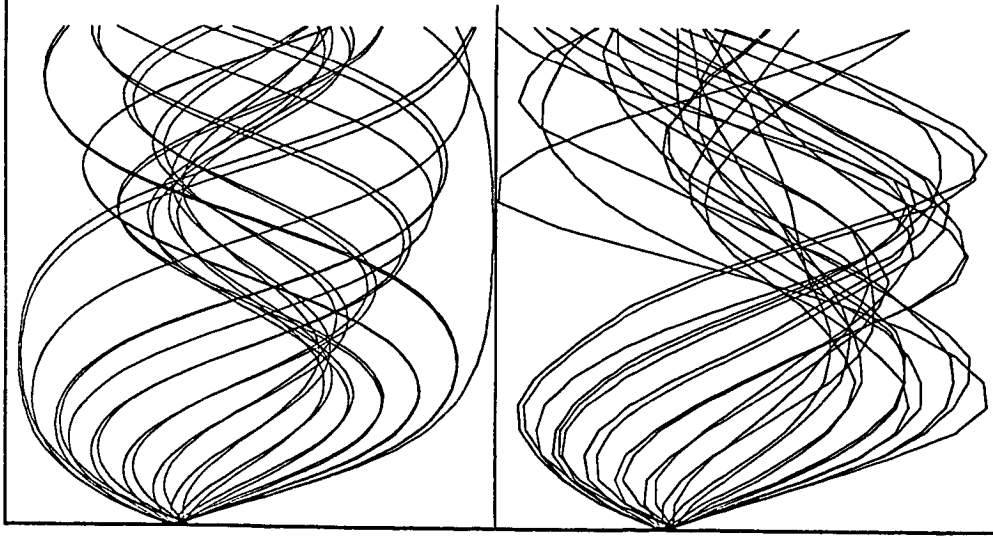

Figure 3.

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